From resolvent bounds to semigroup bounds

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Abstract

The purpose of this note is to revisit the proof of the Gearhardt-Prüss-Hwang-Greiner theorem for a semigroup $S(t)$, following the general idea of the proofs that we have seen in the literature and to get an explicit estimate on $\|S(t)\|$ in terms of bounds on the resolvent of the generator.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $[0,+\infty[\ni t \mapsto S(t) \in \mathcal{L}(\mathcal{H},\mathcal{H})$ be a strongly continuous semigroup with $S(0) = I$. Recall that by the Banach-Steinhaus theorem, $\sup_J \|S(t)\| =: m(J)$ is bounded for every compact interval $J \subset [0,+\infty[$. Using the semigroup property it follows easily that there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $S(t)$ has the property

$$P(M,\omega_0): \quad \|S(t)\| \leq Me^{\omega_0 t},\ t \geq 0. \tag{1.1}$$

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In fact, we have this for $0 \leq t < 1$ and for larger values of t, write $t = [t] + r$, $[t] \in \mathbb{N}$, $0 \leq r < 1$, and $S(t) = S(1)^{[t]} S(r)$.

Let A be the generator of the semigroup (so that formally $S(t) = \exp tA$) and recall (cf. [8], Chapter II or [18]) that A is closed and densely defined. We also recall ([8], Theorem II.1.10) that

$$(z - A)^{-1} = \int_0^\infty S(t) e^{-tz} dt, \quad \|(z - A)^{-1}\| \leq \frac{M}{\operatorname{Re} z - \omega_0}, \tag{1.2}$$

when $P(M, \omega_0)$ holds and z belongs to the open half-plane $\operatorname{Re} z > \omega_0$.

Recall the Hille-Yoshida theorem ([8], Th. II.3.5) according to which the following three statements are equivalent when $\omega \in \mathbb{R}$:

- $P(1, \omega)$ holds.
- $\|(z - A)^{-1}\| \leq (\operatorname{Re} z - \omega)^{-1}$, when $z \in \mathbb{C}$ and $\operatorname{Re} z > \omega$.
- $\|(\lambda - A)^{-1}\| \leq (\lambda - \omega)^{-1}$, when $\lambda \in]\omega, +\infty[$.

Here we may notice that we get from the special case $\omega = 0$ to general ω by passing from $S(t)$ to $\widetilde{S}(t) = e^{-\omega t} S(t)$.

Also recall that there is a similar characterization of the property $P(M, \omega)$ when $M > 1$, in terms of the norms of all powers of the resolvent. This is the Feller-Miyadera-Phillips theorem ([8], Th. II.3.8). Since we need all powers of the resolvent, the practical usefulness of that result is less evident.

We next recall the Gearhardt-Prüss-Hwang-Greiner theorem, see [8], Theorem V.I.11, [24], Theorem 19.1:

Theorem 1.1.
(a) Assume that $\|(z-A)^{-1}\|$ *is uniformly bounded in the half-plane* $\operatorname{Re} z \geq \omega$. *Then there exists a constant* $M > 0$ *such that* $P(M, \omega)$ *holds.*
(b) If $P(M, \omega)$ *holds, then for every* $\alpha > \omega$, $\|(z-A)^{-1}\|$ *is uniformly bounded in the half-plane* $\operatorname{Re} z \geq \alpha$.

The part (b) follows from (1.2) with ω_0 replaced by ω.

The purpose of this note is to revisit the proof of (a), following the general idea of the proofs that we have seen in the literature and to get an explicit t dependent estimate on $e^{-\omega t} \|S(t)\|$, implying explicit bounds on M.

This idea is essentially to use that the resolvent and the inhomogeneous equation $(\partial_t - A)u = w$ in exponentially weighted spaces are related via Fourier-Laplace transform and we can use Plancherel's formula. Variants of this simple idea have also been used in more concrete situations. See [3, 10, 17, 20].

Note that we can improve a little the conclusion of (a). If the property (a) is true for some ω then it is automatically true for some $\omega' < \omega$. We recall indeed the following

Lemma 1.2.
If for some $r(\omega) > 0$, $\|(z-A)^{-1}\| \le \frac{1}{r(\omega)}$ for $\operatorname{Re} z > \omega$, then for every $\omega' \in]\omega - r(\omega), \omega]$ we have

$$\|(z-A)^{-1}\| \le \frac{1}{r(\omega) - (\omega - \omega')}, \quad \operatorname{Re} z > \omega'.$$

Proof. Let $\widetilde{z} \in \mathbb{C}$, $\operatorname{Re} \widetilde{z} > \omega$. Then $\|(\widetilde{z} - A)^{-1}\| \le \frac{1}{r(\omega)}$. For $z \in \mathbb{C}$ with $|z - \widetilde{z}| < r(\omega)$, we have

$(z-A)(\widetilde{z}-A)^{-1} = 1+(z-\widetilde{z})(\widetilde{z}-A)^{-1}$, where $\|(z-\widetilde{z})(\widetilde{z}-A)^{-1}\| \le |z-\widetilde{z}|/r(\omega) < 1$,

so $1 + (z-\widetilde{z})(\widetilde{z}-A)^{-1}$ is invertible and

$$\|(1 + (z-\widetilde{z})(\widetilde{z}-A)^{-1})^{-1}\| \le \frac{1}{1 - |z-\widetilde{z}|/r(\omega)}.$$

Hence z belongs to the resolvent set of A and

$$(z-A)^{-1} = (\widetilde{z}-A)^{-1}(1+(z-\widetilde{z})(\widetilde{z}-A)^{-1})^{-1}, \ \|(z-A)^{-1}\| \le \frac{1}{r(\omega) - |z-\widetilde{z}|}.$$

Now, if $z \in \mathbb{C}$ and $\operatorname{Re} z > \omega'$, we can find $\widetilde{z} \in \mathbb{C}$ with $\operatorname{Re} z > \omega$, $|z - \widetilde{z}| < \omega - \omega'$ and the lemma follows. □

Remark 1.3.
Let

$$\omega_0 = \inf\{\omega \in \mathbb{R}\,\{z \in \mathbb{C};\, \operatorname{Re} z > \omega\} \subset \rho(A) \textit{ and } \sup_{\operatorname{Re} z > \omega} \|(z-A)^{-1}\| < \infty\}.$$

For $\omega > \omega_0$, we may define $r(\omega)$ by

$$\frac{1}{r(\omega)} = \sup_{\operatorname{Re} z > \omega} \|(z-A)^{-1}\|.$$

Then $r(\omega)$ is an increasing function of ω; for every $\omega \in]\omega_0, \infty[$, we have $\omega - r(\omega) \geq \omega_0$ and for $\omega' \in [\omega - r(\omega), \omega]$ we have

$$r(\omega') \geq r(\omega) - (\omega - \omega').$$

We may state all this more elegantly by saying that r is a Lipschitz function on $]\omega_0, +\infty[$ satisfying

$$0 \leq \frac{dr}{d\omega} \leq 1\,.$$

Moreover, if $\omega_0 > -\infty$, then $r(\omega) \to 0$ when $\omega \searrow \omega_0$.

Remark 1.4.
Notice that by (1.1), (1.2), we already know that $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\operatorname{Re} z \geq \beta$, if $\beta > \omega_0$. If $\alpha \leq \omega_0$, we see that $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\operatorname{Re} z \geq \alpha$, provided that

- *we have this uniform boundedness on the line $\operatorname{Re} z = \alpha$,*
- *A has no spectrum in the half-plane $\operatorname{Re} z \geq \alpha$,*
- *$\|(z-A)^{-1}\|$ does not grow too wildly in the strip $\alpha \leq \operatorname{Re} z \leq \beta$: $\|(z-A)^{-1}\| \leq \mathcal{O}(1)\exp(\mathcal{O}(1)\exp(k|\operatorname{Im} z|))$, where $k < \pi/(\beta - \alpha)$.*

We then also have

$$\sup_{\operatorname{Re} z \geq \alpha} \|(z-A)^{-1}\| = \sup_{\operatorname{Re} z = \alpha} \|(z-A)^{-1}\|. \tag{1.3}$$

This follows from the subharmonicity of $\ln ||(z-A)^{-1}||$, Hadamard's theorem (or Phragmén-Lindelöf in exponential coordinates) and the maximum principle.

Our main result is:

Theorem 1.5.
We make the assumptions of Theorem 1.1, (a) and define $r(\omega) > 0$ by

$$\frac{1}{r(\omega)} = \sup_{\operatorname{Re} z \geq \omega} \|(z-A)^{-1}\|.$$

Let $m(t) \geq \|S(t)\|$ be a continuous positive function. Then for all $t, a, \widetilde{a} > 0$, such that $t = a + \widetilde{a}$, we have

$$\|S(t)\| \leq \frac{e^{\omega t}}{r(\omega)\|\frac{1}{m}\|_{e^{-\omega \cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\omega \cdot}L^2([0,\widetilde{a}])}}. \tag{1.4}$$

Here the norms are always the natural ones obtained from $\mathcal{H}$, L^2, thus for instance $\|S(t)\| = \|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$, if u is a function on $\mathbb{R}$ with values in $\mathbb{C}$ or in $\mathcal{H}$, $\|u\|$ denotes the natural L^2 norm, when the norm is taken over a subset J of $\mathbb{R}$, this is indicated with a "$L^2(J)$". In (1.4) we also have the natural norm in the exponentially weighted space $e^{-\omega \cdot}L^2([0,a])$ and similarly with $\widetilde{a}$ instead of a; $\|f\|_{e^{-\omega \cdot}L^2([0,a])} = \|e^{\omega \cdot}f(\cdot)\|_{L^2([0,a])}$.

As we shall see in the next section, under the assumption of the theorem, we have $P(M,\omega)$ with an explicit M. See also the appendix.

We also have the following variant of the main result that can be useful in problems of return to equilibrium.

Theorem 1.6.
We make the assumptions of Theorem 1.5, so that (1.4) holds. Let $\widetilde{\omega} < \omega$ and assume that A has no spectrum on the line $\operatorname{Re} z = \widetilde{\omega}$ and that the spectrum of A in the half-plane $\operatorname{Re} z > \widetilde{\omega}$ is compact (and included in the strip $\widetilde{\omega} < \operatorname{Re} z < \omega$). Assume that $\|(z-A)^{-1}\|$ is uniformly bounded on $\{z \in \mathbb{C};\ \operatorname{Re} z \geq \widetilde{\omega}\} \setminus U$, where U is any neighborhood of $\sigma_+(A) := \{z \in \sigma(A);\ \operatorname{Re} z > \widetilde{\omega}\}$ and define $r(\widetilde{\omega})$ by

$$\frac{1}{r(\widetilde{\omega})} = \sup_{\operatorname{Re} z = \widetilde{\omega}} \|(z-A)^{-1}\|.$$

Then for every $t > 0$,

$$S(t) = S(t)\Pi_+ + R(t) = S(t)\Pi_+ + S(t)(1-\Pi_+),$$

where for all $a, \widetilde{a} > 0$ with $a + \widetilde{a} = t$,

$$\|R(t)\| \leq \frac{e^{\widetilde{\omega} t}}{r(\widetilde{\omega})\|\frac{1}{m}\|_{e^{-\widetilde{\omega} \cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\widetilde{\omega} \cdot}L^2([0,\widetilde{a}])}}\|I - \Pi_+\|. \tag{1.5}$$

Here Π_+ denotes the spectral projection associated to $\sigma_+(A)$:

$$\Pi_+ = \frac{1}{2\pi i}\int_{\partial V} (z-A)^{-1}dz,$$

where V is any compact neighborhood of $\sigma_+(A)$ with C^1 boundary, disjoint from $\sigma(A) \setminus \sigma_+(A)$.

2 Applications : Explicit bounds in the abstract framework

Theorem 1.5 has two ingredients: the existence of some initial control by $m(t)$ and the additional information on the resolvent.

2.1 A quantitative Gearhardt-Prüss statement

As observed in the introduction (see (1.1)), we have at least an estimate with $m(t) = \widehat{M} \exp \widehat{\omega} t$, for some $\widehat{\omega} \geq \omega$. We apply Theorem 1.5 with this $m(t)$ and $a = \tilde{a} = \frac{t}{2}$. The term appearing in the denominator of (1.4) becomes

$$\|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,\tilde{a}])} = \frac{1}{2} \widehat{M}^{-2}\, t\,, \tag{2.1}$$

if $\widehat{\omega} = \omega$, and

$$= \frac{1}{2\widehat{M}^2(\widehat{\omega} - \omega)} \left[1 - \exp((\omega - \widehat{\omega})t)\right]\,, \tag{2.2}$$

if $\widehat{\omega} > \omega$.

Hence we obtain the estimate with a new $m^{new}(t)$, with

$$m^{new}(t) = \frac{2\widehat{M}^2(\widehat{\omega} - \omega)}{r(\omega)[1 - \exp((\omega - \widehat{\omega})t)]}\ \exp \omega t.$$

This gives in particular that $S(t)$ satisfies $P(M, \omega)$, with

$$M = \sup_t \left(\exp -\omega t\ \min(\widehat{M} \exp \widehat{\omega} t, m^{new}(t))\right)\,.$$

We will see how to optimize over ω in Subsection 2.3.

Let us push the computation. Without loss of generality, we can assume $\widehat{\omega} = 0$ and we make the assumption in Theorem 1.5 for some $\omega < 0$. Combining Theorem 1.5 and the trivial estimate

$$||S(t)|| \leq \widehat{M} = \widehat{M} \exp -\omega t\ \exp \omega t$$

we obtain that we have $P(M, \omega)$ with

$$M = \widehat{M} \sup_t \left(\min(\exp -\omega t, \frac{2\widehat{M}|\omega|}{r(\omega)(1 - \exp \omega t)})\right)\,.$$

This can be rewritten in the form:

$$M = \widehat{M} \sup_{u \in]0,1[} \left(\min(\frac{1}{u}, \frac{2\widehat{M}|\omega|}{r(\omega)(1-u)} \right) = 1 + 2\frac{\widehat{M}|\omega|}{r(\omega)}\,.$$

Proposition 2.1.
Let $S(t)$ be a continuous semigroup such that $P(\widehat{M}, \widehat{\omega})$ is satisfied for some pair $(\widehat{M}, \widehat{\omega})$ and such that $r(\omega) > 0$ for some $\omega < \widehat{\omega}$. Then:

$$||S(t)|| \leq \widehat{M} \left(1 + \frac{2\widehat{M}(\widehat{\omega} - \omega)}{r(\omega)} \right) \exp \omega t\,. \tag{2.3}$$

2.2 Estimate with exponential gain.

In the same spirit, and combining with Lemma 1.2, we get the following extension of (2.3) (with $\widehat{\omega} = 0$)

$$||S(t)|| \leq \widehat{M} \left(\frac{(1-s)r(\omega) + 2\widehat{M}(\widehat{\omega} - \omega + sr(\omega))}{(1-s)r(\omega)} \right) \exp(\omega - sr(\omega))t\,,\ \forall s \in [0,1[\,. \tag{2.4}$$

Taking $s = \frac{t}{1+t}$ gives a rather optimal decay at ∞ in $\mathcal{O}(t) \exp(\omega - r(\omega))t$.

If we assume now instead the control of the norm of the resolvent on $\operatorname{Re} z \geq 0$, hence if we are in the case $\omega = \widehat{\omega} = 0$, we get

$$||S(t)|| \leq \frac{2\widehat{M}}{r(0)t}\,,$$

and using the semi-group property $\leq \left(\frac{2\widehat{M}N}{r(0)t}\right)^N$, for any $N \geq 1$. Hence we can get an explicit control of the decay of $S(t)$, by optimizing over N. As in the theory of analytic symbols, we can take $N = E(\alpha t)$ where $E(s)$ denotes the integer part of s and α such that $\alpha < r(0)/(2\widehat{M})$, we get an exponential decay of $S(t)$.
Alternately, we can use the extension of the resolvent on $\operatorname{Re} z > -sr(0)$ and this leads to :

$$||S(t)|| \leq \widehat{M} \left(\frac{(1-s) + 2\widehat{M}s}{(1-s)} \right) \exp(-sr(0))t\,,\ \forall s \in [0,1[\,. \tag{2.5}$$

2.3 The limit $\omega \searrow \omega_0$

Consider the situation of Theorem 1.5 and let ω_0 be as in Remark 1.3. Assume that $\omega_0 > -\infty$ so that $r(\omega) \to 0$, when $\omega \to \omega_0$. For $t \geq 1$, $\omega > \omega_0$, we get from (1.4):

$$e^{-\omega_0 t}\|S(t)\| \leq \frac{e^{t(\omega-\omega_0)}}{r(\omega)\int_0^{1/2} m(s)^{-2}e^{2\omega_0 s}ds} \leq \mathcal{O}(1)\frac{e^{t(\omega-\omega_0)}}{r(\omega)}. \tag{2.6}$$

Optimizing over $\omega \in]\omega_0, \omega_0+\epsilon_0]$, we get the existence of C such that

$$e^{-\omega_0 t}\|S(t)\| \leq C \exp \Phi(t)\,, \tag{2.7}$$

with

$$\Phi(t) = \inf_{\omega\in]\omega_0,\omega_0+\epsilon_0]} t(\omega-\omega_0) - \ln r(\omega)\,.$$

It is clear that $\lim_{t\to+\infty}\Phi(t)/t = 0$, but to have a more quantitative version, we need some information on the behavior of $r(\omega)$ as $\omega \searrow \omega_0$. Let us treat two examples.

If

$$r(\omega) \geq \frac{(\omega-\omega_0)^k}{C},\ \text{when } 0 < \omega-\omega_0 \ll 1,$$

for some constants $C, k > 0$, then choosing $\omega - \omega_0 = k/t$ in (2.6), we get

$$e^{-\omega_0 t}\|S(t)\| \leq \mathcal{O}(1)t^k,\ t \geq 1.$$

On the other hand, if

$$r(\omega) \geq \exp -\frac{(\omega-\omega_0)^{-\alpha}}{C\alpha},\ \text{when } 0 < \omega-\omega_0 \ll 1,$$

for some constants $C, \alpha > 0$, then

$$\frac{e^{t(\omega-\omega_0)}}{r(\omega)} \leq \exp\left(t(\omega-\omega_0) + \frac{(\omega-\omega_0)^{-\alpha}}{C\alpha}\right),$$

and choosing $\omega - \omega_0 = (Ct)^{-\frac{1}{\alpha+1}}$, we get the existence of a constant $\widehat{C}$ such that

$$e^{-\omega_0 t}\|S(t)\| \leq e^{\widehat{C}t^{\frac{\alpha}{\alpha+1}}},\ t \geq 1.$$

3 Applications to concrete examples

3.1 The complex Airy operator on the half-line

Let us consider (as in [1]) the Dirichlet realization P^D of the Airy operator on $\mathbb{R}^+$: $D_x^2 + ix$ and P the realization of $D_x^2 + ix$ in $\mathbb{R}$. One can determine explicitly its spectrum (using Sibuya's theory or Combes-Thomas's trick) as

$$\sigma(P^D) := \{\lambda_j \, e^{i\frac{\pi}{3}} \, , \, j \in \mathbb{N}^*\}$$

where the λ_j's are the eigenvalues (immediately related to the zeroes of the Airy function) of the Dirichlet realization in $\mathbb{R}^+$ of $D_x^2 + x$.

It was shown in [11], that $||(P^D - z)^{-1}||$ is as $\operatorname{Re} z > 0$ and $\operatorname{Im} z \mapsto +\infty$ asymptotically equivalent to $||(P - \operatorname{Re} z)^{-1}||$ and that $||(P^D - z)^{-1}||$ tends to 0 as $\operatorname{Re} z > 0$ and $\operatorname{Im} z \mapsto -\infty$. The standard Gearhardt-Prüss theorem, applied to $A := -P^D$, permits to show that, for any $\omega > -\lambda_1 \cos\frac{\pi}{3}$, we have

$$||S(t)|| \leq M_\omega \exp(\omega t) \, .$$

Theorem 1.6 permits the following improvment :

$$S(t) = \exp\left(-e^{i\frac{\pi}{3}} \, \lambda_1 \, t\right) \, \Pi_+ + R(t) \, ,$$

with

$$||R(t)|| \leq M_{\tilde{\omega}} \exp(\tilde{\omega} t) \, ,$$

for any $\tilde{\omega} > -\lambda_2 \cos\frac{\pi}{3}$.
Here Π_+ is the projector associated with the eigenfunction of P^D associated with $\lambda_1 \, e^{i\frac{\pi}{3}}$. Hence we get a much better control of the semi-group.

3.2 The case of the Kramers-Fokker-Planck operator

Inspired by the work by F. Hérau and F. Nier [15], F. Hérau, J. Sjöstrand and C. Stolk [16] studied the Kramers-Fokker-Planck operator

$$P = y \cdot h\partial_x - V'(x) \cdot h\partial_x + \frac{\gamma}{2}(y - h\partial_y)(y + h\partial_y) \tag{3.1}$$

on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, where $\gamma > 0$ is fixed and we let $h \to 0$. We assume that $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ with $\partial^\alpha V = \mathcal{O}(1)$ for every $\alpha \in \mathbb{N}^n$ of length ≥ 2 and we also

assume that V is a Morse function such that $|\nabla V(x)| \geq 1/C$ when $|x| \geq C$ for some constant $C > 0$. Then we know from [15] and under much weaker assumptions from B. Helffer, F. Nier [12] that P is maximally accretive with $\operatorname{Re} P \geq 0$, so that P generates a semi-group of contractions $e^{-tP/h}$, $t \geq 0$. In particular the spectrum of P is contained in the closed right half plane. In [16] it was shown that for every fixed $C > 0$ and for $h > 0$ small enough, the spectrum of P in the strip $0 \leq \operatorname{Re} z \leq Ch$ is discrete and the eigenvalues are of the form

$$E_j = \lambda_j h + o(h), \quad \operatorname{Re} \lambda_j \leq Ch, \tag{3.2}$$

where λ_j are eigenvalues of the different quadratic approximations of $P_{h=1}$ at the various points $(x_k, 0)$ where $V'(x_k) = 0$. Here the points E_j all belong to a sector $|\operatorname{Im} \lambda| \leq \mathcal{O}(\operatorname{Re} \lambda)$, so the eigenvalues in (3.2) are all confined to a disc $D(0, \widetilde{C}h)$.

It was also shown in [16] that if $\widetilde{\omega} \geq 0$ and $\operatorname{Re} \lambda_j \neq \widetilde{\omega}$ for all the eigenvalues λ_j, then $\|(P - z)^{-1}\| = \mathcal{O}(1/h)$ uniformly on the line $\operatorname{Re} z = h\widetilde{\omega}$. The same estimate holds when $0 \leq \operatorname{Re} z \leq Ch$ and $|z| \geq \widetilde{C}h$. Actually, using a form of semi-classical sub-ellipticity (closely related in spirit to the one established in [15] and further studied in [12]) it was also shown that this estimate holds in a larger parabolic neighborhood of $i\mathbb{R}$ away from the disc $D(0, \widetilde{C}h)$, and using this stronger result and a contour deformation in a standard integral representation of $e^{-tP/h}$ (again in the spirit of [15]) it was established in [16] that

$$e^{-tP/h} = e^{-tP/h}\Pi_+ + R(t), \tag{3.3}$$

where Π_+ is the spectral projection associated with $\{z \in \sigma(P); 0 \leq \operatorname{Re} z \leq \widetilde{\omega}\}$, and $\|R(t)\| \leq \mathrm{Const.}\, e^{-t\widetilde{\omega}}$. Now this result becomes a direct application of Theorem 1.6 to $A := -P/h$ and we do not need any bounds on the resolvent in the region $\operatorname{Re} z > h\widetilde{\omega}$.

In [13, 14] similar results were obtained for more general operators, for which we do not necessarily have any bound on the resolvent beyond a strip, and the proof was to use microlocal coercivity outside a compact set in slightly weighted L^2-spaces. Again Theorem 1.6 would give some simplifications.

3.3 The complex harmonic oscillator

The complex harmonic oscillator

$$P := D_x^2 + ix^2$$

on the line was studied by E.B. Davies [4, 5], L. Boulton, [2] and M. Zworski [26] in connection with the analysis of the pseudospectra. As for the complex Airy operator, it is easy to determine the spectrum which is given by $e^{i\frac{\pi}{4}}(2j+1)$, $j \in \mathbb{N}$. This operator is maximally accretive and we can apply Theorem 1.6 with $A = -P$. From these works as well as those of K. Pravda Starov [19] and Dencker-Sjöstrand-Zworski [7], we know that for fixed $\operatorname{Re} z$ as $\operatorname{Im} z \to +\infty$,

$$\lim_{\operatorname{Im} z \to +\infty} ||(P-z)^{-1}|| = 0\,.$$

More precisely, for any compact interval K, there exists $C > 0$ such that

$$||(P-z)^{-1}|| \leq C\,|\operatorname{Im} z|^{-\frac{1}{3}}\,, \text{ for } \operatorname{Im} z \geq C,\ \operatorname{Re} z \in K\,.$$

This follows from [19, 7], notice here that the results in [7] are given in the semi-classical limit for the spectral parameter in a compact set, but there is a simple scaling argument, allowing to pass to the limit of high frequency. See for example [21, 22]. As $\operatorname{Im} z \to -\infty$ we have by more elementary estimates:

$$||(P-z)^{-1}|| \leq \ |\operatorname{Im} z|^{-1}\,, \text{ for } \operatorname{Im} z < 0\,.$$

We can therefore apply Theorem 1.6 and get

$$S(t) = \exp\left(-e^{i\frac{\pi}{4}}\, t\right)\ \Pi_+ + R(t)\,,$$

with

$$||R(t)|| \leq M_{\tilde{\omega}} \exp(\tilde{\omega} t)\,,$$

for any $\tilde{\omega} > -3\,\cos\frac{\pi}{4}$. Here Π_+ is the spectral projection associated with the eigenvalue $e^{i\frac{\pi}{4}}$ of P.

Hence we get again a much better control of the semi-group.

4 Proofs of the main statements

4.1 Proof of Theorem 1.5

As already mentioned, we shall use the inhomogeneous equation

$$(\partial_t - A)u = w \text{ on } \mathbb{R}. \tag{4.1}$$

Recall that if $v \in \mathcal{H}$, then $S(t)v \in C^0([0,\infty[;\mathcal{H})$, while if $v \in \mathcal{D}(A)$, then $S(t)v \in C^1([0,\infty[;\mathcal{H}) \cap C^0([0,\infty[;\mathcal{D}(A))$ and

$$AS(t)v = S(t)Av, \quad (\partial_t - A)S(t)v = 0\,. \tag{4.2}$$

Let $C^0_+(\mathcal{H})$ denote the subspace of all $v \in C^0(\mathbb{R};\mathcal{H})$ that vanish near $-\infty$. For $k \in \mathbb{N}$, we define $C^k_+(\mathcal{H})$ and $C^k_+(\mathcal{D}(A))$ similarly. For $w \in C^0_+(\mathcal{H})$, we define $Ew \in C^0_+(\mathcal{H})$ by

$$Ew(t) = \int_{-\infty}^{t} S(t-s)w(s)ds. \tag{4.3}$$

It is easy to see that E is continuous: $C^k_+(\mathcal{H}) \to C^k_+(\mathcal{H})$, $C^k_+(\mathcal{D}(A)) \to C^k_+(\mathcal{D}(A))$ and if $w \in C^1_+(\mathcal{H}) \cap C^0_+(\mathcal{D}(A))$, then $u = Ew$ is the unique solution in the same space of (4.1). More precisely, we have

$$(\partial_t - A)Ew = w, \quad E(\partial_t - A)u = u, \tag{4.4}$$

for all $u, w \in C^1_+(\mathcal{H}) \cap C^0_+(\mathcal{D}(A))$

Now recall that we have $P(M,\omega_0)$ in (1.1) for some M,ω_0. If $\omega_1 > \omega_0$ and $w \in C^0_+(\mathcal{H}) \cap e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})$ (by which we only mean that $w \in C^0_+(\mathcal{H})$ and that $\|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} < \infty$, avoiding to define the larger space $e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})$), then Ew belongs to the same space and

$$\begin{aligned}
\|Ew\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} &\leq \left(\int_0^\infty e^{-\omega_1 t}\|S(t)\|\,dt\right) \|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})} \\
&\leq \frac{M}{\omega_1 - \omega_0}\|w\|_{e^{\omega_1 \cdot} L^2(\mathbb{R};\mathcal{H})}.
\end{aligned}$$

Now we consider Laplace transforms. If $u \in e^{\omega \cdot}\mathcal{S}(\mathbb{R};\mathcal{H})$, then the Laplace transform

$$\widehat{u}(\tau) = \int_{-\infty}^{+\infty} e^{-t\tau}u(t)dt$$

is well-defined in $\mathcal{S}(\Gamma_\omega;\mathcal{H})$, where

$$\Gamma_\omega = \{\tau \in \mathbb{C};\, \operatorname{Re}\tau = \omega\}$$

and we have Parseval's identity

$$\frac{1}{2\pi}\|\widehat{u}\|^2_{L^2(\Gamma_\omega)} = \|u\|_{e^{\omega \cdot} L^2}. \tag{4.5}$$

Now we make the assumptions in Theorem 1.5, define ω and $r(\omega)$ as there, and let M, ω_0 be as above. Let $w \in e^{\omega\cdot}\mathcal{S}_+(\mathcal{D}(A))$, where $\mathcal{S}_+(\mathcal{D}(A))$ by definition is the space of all $u \in \mathcal{S}(\mathbb{R}; \mathcal{D}(A))$, vanishing near $-\infty$. Then $w \in e^{\omega_1\cdot}\mathcal{S}_+(\mathcal{D}(A))$ for all $\omega_1 \geq \omega$. If $\omega_1 > \omega_0$ then $u := Ew$ belongs to $e^{\omega_1\cdot}\mathcal{S}_+(\mathcal{D}(A))$ and solves (4.1). Laplace transforming that equation, we get

$$(\tau - A)\widehat{u}(\tau) = \widehat{w}(\tau), \tag{4.6}$$

for $\operatorname{Re}\tau > \omega_0$. Notice here that $\widehat{w}(\tau)$ is continuous in the half-plane $\operatorname{Re}\tau \geq \omega$, holomorphic in $\operatorname{Re}\tau > \omega$, and $\widehat{w}_{|\Gamma_{\widetilde{\omega}}} \in \mathcal{S}(\Gamma_{\widetilde{\omega}})$ for every $\widetilde{\omega} \geq \omega$. We use the assumption in the theorem to write

$$\widehat{u}(\tau) = (\tau - A)^{-1}\widehat{w}(\tau), \tag{4.7}$$

and to see that $\widehat{u}(\tau)$ can be extended to the half-plane $\operatorname{Re}\tau \geq \omega$ with the same properties as $\widehat{w}(\tau)$. By Laplace (Fourier) inversion from Γ_ω we conclude that $u \in e^{\omega\cdot}\mathcal{S}_+(\mathcal{D}(A))$. Moreover, since

$$\|\widehat{u}(\tau)\|_{\mathcal{H}} \leq \frac{1}{r(\omega)}\|\widehat{w}(\tau)\|_{\mathcal{H}},\ \tau \in \Gamma_\omega,$$

we get from Parseval's identity that

$$\|u\|_{e^{\omega\cdot}L^2} \leq \frac{1}{r(\omega)}\|w\|_{e^{\omega\cdot}L^2}. \tag{4.8}$$

Using the density of $\mathcal{D}(A)$ in $\mathcal{H}$ together with standard cutoff and regularization arguments, we see that (4.8) extends to the case when $w \in e^{\omega\cdot}L^2(\mathbb{R}; \mathcal{H}) \cap C^0_+(\mathcal{H})$, leading to the fact that $u := Ew$ belongs to the same space and satisfies (4.8).

Consider $u(t) = S(t)v$, for $v \in D(A)$, solving the Cauchy problem

$$\begin{aligned} &(\partial_t - A)u = 0\,,\ t \geq 0\ , \\ &u(0) = v\,. \end{aligned}$$

Let χ be a decreasing Lipschitz function on $\mathbb{R}$, equal to 1 on $]-\infty, 0]$ and vanishing near $+\infty$. Then

$$(\partial_t - A)(1-\chi)u = -\chi'(t)u\,,$$

and

$$\begin{aligned}\|\chi' u\|^2_{e^{\omega\cdot}L^2} &= \int_0^{+\infty} |\chi'(t)|^2 \|u(t)\|^2 e^{-2\omega t}\,dt \\ &\le \|\chi' m\|^2_{e^{\omega\cdot}L^2}\,\|v\|^2,\end{aligned}$$

where we notice that $\chi' m$ is welldefined on $\mathbb{R}$ since $\operatorname{supp}\chi' \subset [0,\infty[$.

Now $(1-\chi)u$, $\chi' u$ are well-defined on $\mathbb{R}$, so

$$\|(1-\chi)u\|_{e^{\omega\cdot}L^2} \le r(\omega)^{-1}\|\chi' u\|_{e^{\omega\cdot}L^2} \le r(\omega)^{-1}\|\chi' m\|_{e^{\omega\cdot}L^2}\,\|v\|\,. \tag{4.9}$$

Strictly speaking, in order to apply (4.8), we approximate χ by a sequence of smooth functions. Similarly,

$$\|\chi u\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)} \le \|\chi m\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)}\,\|v\|\,,$$

so

$$\|u\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)} \le \left(r(\omega)^{-1}\|\chi' m\|_{e^{\omega\cdot}L^2} + \|\chi m\|_{e^{\omega\cdot}L^2(\mathbb{R}_+)}\right)\|v\|\,.$$

Let us now go from L^2 to L^∞. For $t>0$, let $\chi_+(s) = \widetilde{\chi}(t-s)$ with $\widetilde{\chi}$ as χ above and in addition $\operatorname{supp}\widetilde{\chi} \subset]-\infty, t]$, so that $\chi_+(t) = 1$ and $\operatorname{supp}\chi_+ \subset [0,\infty[$. Then

$$(\partial_s - A)\left(\chi_+(s)u(s)\right) = \chi_+'(s)u(s)\,,$$

and

$$\chi_+ u(t) = \int_{-\infty}^t S(t-s)\,\chi_+'(s)\,u(s)\,ds\,.$$

Hence, we obtain

$$\begin{aligned} e^{-\omega t}\|u(t)\| &= e^{-\omega t}\|\chi_+(t)u(t)\| \\ &\le \textstyle\int_{-\infty}^t e^{-\omega t}\, m(t-s)|\widetilde{\chi}'(t-s)|\|u(s)\|\,ds \\ &\le \textstyle\int_{-\infty}^t e^{-\omega(t-s)}\, m(t-s)\,|\widetilde{\chi}'(t-s)|\, e^{-\omega s}\|u(s)\|\,ds \\ &\le \|m\widetilde{\chi}'\|_{e^{\omega\cdot}L^2}\,\|u\|_{e^{\omega\cdot}L^2(\operatorname{supp}\chi_+)}\,. \end{aligned} \tag{4.10}$$

Assume that

$$\chi = 0 \text{ on } \operatorname{supp}\chi_+\,. \tag{4.11}$$

Then u can be replaced by $(1-\chi)u$ in the last line in (4.10) and using (4.9) we get

$$e^{-\omega t}\|u(t)\| \le r(\omega)^{-1}\|m\chi'\|_{e^{\omega\cdot}L^2}\|m\widetilde{\chi}'\|_{e^{\omega\cdot}L^2}\|v\|\,. \tag{4.12}$$

Let

$$\operatorname{supp}\chi \subset]-\infty, a]\,,\ \operatorname{supp}\widetilde{\chi} \subset]-\infty, \widetilde{a}], a+\widetilde{a}=t\,, \tag{4.13}$$

so that (4.11) holds.

For a given $a>0$, we look for χ in (4.13) such that $\|m\chi'\|_{e^{\omega\cdot}L^2}$ is as small as possible. By the Cauchy-Schwarz inequality,

$$1=\int_0^a |\chi'(s)|ds \leq \|\chi' m\|_{e^{\omega\cdot}L^2}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(]0,a[)}\,, \tag{4.14}$$

so

$$\|\chi' m\|_{e^{\omega\cdot}L^2} \geq \frac{1}{\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2(]0,a[)}}\,. \tag{4.15}$$

We get equality in (4.15) if for some constant C,

$$|\chi'(s)|m(s)e^{-\omega s}=C\frac{1}{m(s)}e^{\omega s},\ \text{ on } [0,a],$$

i.e.

$$\chi'(s)m(s)e^{-\omega s}=-C\frac{1}{m(s)}e^{\omega s},\ \text{ on } [0,a],$$

where C is determined by the condition $1=\int_0^a|\chi'(s)|ds$.
We get

$$C=\frac{1}{\|\frac{1}{m}\|^2_{e^{-\omega\cdot}L^2(]0,a[)}}\,,$$

Here $\chi(s)=1$ for $s\leq 0$, $\chi(s)=0$ for $s\geq a$,

$$\chi(s)=C\int_s^a \frac{1}{m(\sigma)^2}e^{2\omega\sigma}d\sigma,\ 0\leq s\leq a.$$

With the similar optimal choice of $\widetilde{\chi}$, for which

$$\|\widetilde{\chi}' m\|_{e^{\omega\cdot}L^2}=\frac{1}{\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,\widetilde{a}])}},$$

we get from (4.12):

$$e^{-\omega t}\|u(t)\| \leq \frac{\|v\|}{r(\omega)\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,\widetilde{a}])}}, \tag{4.16}$$

provided that $a,\widetilde{a}>0$, $a+\widetilde{a}=t$, for any $v\in D(A)$. Observing that $D(A)$ is dense in $\mathcal{H}$, this completes the proof of Theorem 1.5.

4.2 Proof of Theorem 1.6

We can apply Theorem 1.5 to the restriction $\widetilde{S}(t)$ of $S(t)$ to the range $\mathcal{R}(1-\Pi_+)$ of $1-\Pi_+$. The generator is the restriction $\widetilde{A}$ of A so we get

$$\|\widetilde{S}(t)\| \le \frac{e^{\widetilde{\omega}t}}{r(\widetilde{\omega})\|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot}L^2([0,\widetilde{a}])}}. \tag{4.17}$$

Then (1.5) follows from the fact that $R(t)=\widetilde{S}(t)(1-\Pi_+)$.

A An iterative improvement of Theorem 1.5

Working entirely on the semi-group side and applying Theorem 1.5 repeatedly, we shall see how to gain an extra decay $\mathcal{O}(1)\exp(-t^{1/2}/C)$ for some $C>0$. It is not clear that this result is of practical use, especially in view of Lemma 1.2, but the computations are amusing.

Recall that under the assumptions of Theorem 1.5 we have the estimate (1.4). Here we may have m bounded continuous for $0\le t<T$ and equal to $+\infty$ for $t\ge T$, where $T>0$.

Write $m(t)=\widetilde{m}(t)e^{\omega t}$. Then (1.4) shows that $\|S(t)\|\le \widehat{m}(t)e^{\omega t}$, where

$$\widehat{m}(t)\le \frac{1}{r(\omega)\|\frac{1}{\widetilde{m}}\|_{[0,a]}\|\frac{1}{\widetilde{m}}\|_{[0,\widetilde{a}]}},\ a+\widetilde{a}=t. \tag{A.1}$$

Take $a=\widetilde{a}=t/2$ and divide the previous inequality by $r(\omega)$:

$$\frac{\widehat{m}(t)}{r(\omega)}\le \frac{1}{\int_0^{t/2}(\frac{r(\omega)}{\widetilde{m}(s)})^2\,ds},$$

which we can also write

$$\widehat{f}(t)\ge \int_0^{t/2}\widetilde{f}(s)^2\,ds\,,\ \widetilde{f}(t):=\frac{r(\omega)}{\widetilde{m}(t)},\ \widehat{f}(t):=\frac{r(\omega)}{\widehat{m}(t)}\,.$$

Now assume that $e^{-\omega t}\|S(t)\|\le \widetilde{m}(t)\le \mathcal{O}(1)$ for $0\le t<T$. Then we extend $\widetilde{m}$ to $[0,+\infty[$, by defining

$$\frac{\widetilde{m}(t)}{r(\omega)}=\frac{1}{\int_0^{t/2}(\frac{r(\omega)}{\widetilde{m}(s)})^2\,ds}, \tag{A.2}$$

first for $T \leq t < 2T$, then for $2T \leq t < 4T$ and so on. Correspondingly, we have

$$\widetilde{f}(t) = \int_0^{t/2} \widetilde{f}(s)^2 \, ds, \ t \geq T. \tag{A.3}$$

Theorem 1.5 now shows that $e^{-\omega t}\|S(t)\| \leq \widetilde{m}(t) \leq \mathcal{O}(1)$ for all $t \geq 0$. By construction we see that $\widetilde{m}(t)$ is decreasing on $[T, +\infty[$, so we have

$$e^{-\omega t}\|S(t)\| \leq M, \ \ M = \max(\sup_{[0,T[} \widetilde{m}, \frac{1}{r(\omega)\int_0^{T/2} \widetilde{m}(s)^{-2} \, ds}). \tag{A.4}$$

Notice that $\widetilde{f}$ is increasing on $[T, +\infty[$. We look for upper bounds on $\widetilde{m}$ or equivalently for lower bounds on $\widetilde{f}$. For $k \geq 1$, put $I_k = [T2^{k-1}, T2^k[$, so that the length of I_k is $|I_k| = T2^{k-1}$. Put

$$F(k) = \inf_{I_k} \widetilde{f} = \widetilde{f}(T2^{k-1}) \text{ when } k \geq 1, \quad F(0) = \inf_{[0,T[} \widetilde{f}(t) \, .$$

Then, $F(1) = \int_0^{T/2} \widetilde{f}(t)^2 \, dt \geq \frac{T}{2}F(0)^2$, which we write

$$TF(1) \geq \frac{1}{2}(TF(0))^2 \, .$$

For $k \geq 1$, we get

$$F(k+1) \geq \int_0^{T2^{k-1}} \widetilde{f}(t)^2 dt \geq TF(0)^2 + TF(1)^2 + 2TF(2)^2 + ... + 2^{k-2}TF(k-1)^2,$$

which we write

$$TF(k+1) \geq (TF(0))^2 + (TF(1))^2 + 2(TF(2))^2 + ... + 2^{k-2}(TF(k-1))^2. \tag{A.5}$$

Since $\widetilde{f}$ is increasing on $[T, +\infty[$, we have

$$F(1) \leq F(2) \leq F(3) \leq ...$$

Thus for $k \geq 2$,

$$TF(k+1) \geq 2^{k-2}(TF(k-1))^2 \geq 2^{k-2}(TF(1))^2 \geq 2^{k-4}(TF(0))^4 \, ,$$

which we write

$$TF(k) \geq 2^{k-5}(TF(0))^4 \, , \ k \geq 3.$$

Let k_0 be the smallest integer $k \geq 3$ such that

$$2^{k-5}(TF(0))^4 \geq 2,$$

so that $TF(k) \geq 2$ for $k \geq k_0$.

Now return to (A.5) which implies that

$$TF(k+1) \geq 2^{k-2}(TF(k-1))^2,\ k \geq 1.$$

We get

$$TF(k+2) \geq 2^{k-1}(TF(k))^2,\ k \geq 1,$$

implying,

$$T(F(k+2)) \geq (TF(k))^2, \quad \ln(TF(k+2)) \geq 2\ln(TF(k)).$$

In particular,

$$\ln(TF(k_0+2\nu)) \geq 2^\nu \ln(TF(k_0)) \geq 2^\nu \ln 2,\ \nu \in \mathbb{N}.$$

We conclude that

$$T\widetilde{f}(t) \geq 2^{2^\nu},\ 2^{k_0+2\nu-1} \leq t/T < 2^{k_0+2\nu}.$$

The last inequality for t implies that $2^\nu > (2^{-k_0}t/T)^{1/2}$, so we get

$$T\widetilde{f}(t) \geq 2^{(2^{-k_0}t/T)^{1/2}},\ t/T \geq 2^{k_0-1}, \tag{A.6}$$

or equivalently,

$$\frac{\widetilde{m}(t)}{r(\omega)T} \leq 2^{-(2^{-k_0}t/T)^{1/2}},\ t/T \geq 2^{k_0-1}, \tag{A.7}$$

where we recall that k_0 is the smallest integer such that

$$2^{k_0} \geq \max(\frac{2^6}{(TF(0))^4}, 8). \tag{A.8}$$

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